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COMMENT

Perimeter polynomials for bond percolation processes

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Abstract. Perimeter polynomials are given for the bond percolation problem on the following lattices: the triangular up to D_9, simple quadratic (D_{13}), honeycomb (D_{17}), face-centred cubic (D_7), body-centred cubic (D_8), simple cubic (D_9) and diamond (D_{13}). The total number of bond clusters grouped by size is given to two further orders in each case.

An area of lattice statistics which has received considerable attention during the last decade is the percolation problem and the associated problem of lattice animals, i.e. the total number of connected clusters of size n. The rapid advances have been kept under review by a number of authors, e.g. Shante and Kirkpatrick (1971), Essam (1971, 1972, 1980), Kirkpatrick (1973), de Gennes (1976), Welsh (1977), Wu (1978) and Stauffer (1979).

In this Comment we report perimeter polynomials for the bond percolation problem on the standard lattices in two and three dimensions. Following Sykes and Glen (1976), we define the perimeter polynomials $D_n(q)$ through the relation

$$K(p, q) = \sum_{n=1}^{\infty} D_n(q)p^n,$$

where $K(p, q)$ is the mean number of finite bond clusters per lattice site at density p $(=q-1)$. The polynomial $D_n(q)$ summarises the average environmental situation for all clusters with n bonds. The polynomials, which we give in the appendix, have generally been obtained by direct computer enumeration and we are indebted to J L Martin for assistance at all stages of the work. The first eight polynomials for the simple quadratic and simple cubic lattices are consistent with those derived by Gaunt and Ruskin (1978) for the general d-dimensional simple hypercubic lattice. Perimeter polynomials for the analogous *site* percolation problem have been given by Sykes and Glen (1976) for two-dimensional lattices and by Sykes *et al* (1976) for three-dimensional lattices.

Our own application of the bond perimeter polynomials has been mainly to the derivation of exact series expansions for a study of the critical exponents γ and δ describing, respectively, the asymptotic behaviour of the mean size of finite clusters just below the critical density p_c (Sykes and Glen 1976, Sykes *et al* 1976) and the percolation probability in a small fictitious field at p_c (Gaunt and Sykes 1976, Gaunt 1977a, b).

The total number of n-bond clusters (or bond animals) per lattice site can be obtained to two further orders by using the method described by Sykes and Glen (1976)

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and we give these in tables 1 and 2. The data for the simple quadratic and simple cubic lattices have been reported previously by Gaunt and Ruskin (1978). Some similar data for site animals have been given by Sykes and Glen (1976) and Gaunt *et al* (1976). We have used these data to study and estimate the connective constant λ for bond and site animals, and the exponent τ defined by

$$a_n \sim n^{-\tau}\lambda^n \qquad (n \to \infty)$$

Table 1. Total number of connected bond clusters grouped by bond size. Two-dimensional lattices.

Bonds	Triangular	Simple quadratic	Honeycomb
1	3	2	$1\frac{1}{2}$
2	15	6	3
3	91	22	7
4	603	88	18
5	4 215	372	$49\frac{1}{2}$
6	30 535	1 628	$140\frac{1}{2}$
7	226 905	7 312	408
8	1 718 454	33 466	$1\,207\frac{1}{2}$
9	13 207 569	155 446	3 630
10	102 707 301	730 534	11 037
11	806 366 139	3 466 170	$33\,871\frac{1}{2}$
12		16 576 874	104 776
13		79 810 756	326 283
14		386 458 826	1 021 836
15		1 880 580 352	$3\,215\,857\frac{1}{2}$
16			10 164 252
17			32 247 339
18			$102\,651\,768\frac{1}{2}$
19			$327\,746\,848\frac{1}{2}$

Table 2. Total number of connected bond clusters grouped by bond size. Three-dimensional lattices.

Bonds	Face-centred	Body-centred	Simple cubic	Diamond
1	6	4	3	2
2	66	28	15	6
3	930	252	95	22
4	14 898	2 582	681	91
5	258 252	28 648	5 277	408
6	4 721 698	335 272	43 086	1 928
7	89 700 720	4 077 228	365 313	9 458
8	1 753 975 176	51 033 970	3 186 444	47 718
9	35 076 308 534	653 295 948	28 414 802	246 062
10		8 514 641 328	257 908 020	1 291 128
11			2 375 037 477	6 871 914
12				37 011 338
13				201 346 004
14				1 104 781 410
15				6 107 095 706

where a_n is the number of n-site (or -bond) animals (Sykes and Glen 1976, Sykes *et al* 1976, Guttmann and Gaunt 1978, Gaunt 1980). Whittington and Gaunt (1978) have established rigorous lower bounds for the connective constants for bond and site animals, which establish that they are strictly greater than the corresponding connective constants for self-avoiding walks.

Appendix

A.1. Triangular

$D_1 = 3q^{10}$

$D_2 = 9q^{14} + 6q^{13}$

$D_3 = 29q^{18} + 36q^{17} + 24q^{16} + 2q^{12}$

$D_4 = 99q^{22} + 180q^{21} + 171q^{20} + 126q^{19} + 12q^{16} + 15q^{15}$

$D_5 = 348q^{26} + 846q^{25} + 1\,068q^{24} + 900q^{23} + 768q^{22} + 60q^{20} + 108q^{19} + 114q^{18} + 3q^{14}$

$$\begin{aligned} D_6 = {} & 1\,260q^{30} + 3\,762q^{29} + 6\,186q^{28} + 6\,084q^{27} + 5\,886q^{26} + 4\,284q^{25} \\ & + 601q^{24} + 681q^{23} + 822q^{22} + 906q^{21} + 21q^{18} + 42q^{17} \end{aligned}$$

$$\begin{aligned} D_7 = {} & 4\,644q^{34} + 16\,530q^{33} + 32\,388q^{32} + 39\,924q^{31} + 41\,598q^{30} \\ & + 36\,072q^{29} + 26\,610q^{28} + 8\,850q^{27} + 5\,628q^{26} + 7\,002q^{25} \\ & + 6\,216q^{24} + 534q^{23} + 129q^{22} + 300q^{21} + 474q^{20} + 6q^{16} \end{aligned}$$

$$\begin{aligned} D_8 = {} & 17\,382q^{38} + 71\,598q^{37} + 163\,908q^{36} + 234\,846q^{35} + 285\,426q^{34} \\ & + 276\,918q^{33} + 237\,054q^{32} + 170\,499q^{31} + 93\,801q^{30} + 50\,049q^{29} \\ & + 52\,794q^{28} + 43\,587q^{27} + 10\,140q^{26} + 2\,028q^{25} + 3\,651q^{24} \\ & + 4\,170q^{23} + 435q^{22} + 42q^{20} + 126q^{19} \end{aligned}$$

$$\begin{aligned} D_9 = {} & 65\,822q^{42} + 308\,136q^{41} + 800\,961q^{40} + 1\,330\,992q^{39} + 1\,775\,874q^{38} \\ & + 2\,002\,806q^{37} + 1\,948\,484q^{36} + 1\,535\,682q^{35} + 1\,235\,994q^{34} \\ & + 810\,072q^{33} + 445\,176q^{32} + 401\,634q^{31} + 300\,335q^{30} + 136\,758q^{29} \\ & + 25\,980q^{28} + 35\,388q^{27} + 35\,505q^{26} + 8\,928q^{25} + 276q^{24} \\ & + 960q^{23} + 1\,578q^{22} + 214q^{21} + 14q^{18}. \end{aligned}$$

A.2. Simple quadratic

$D_1 = 2q^6$

$D_2 = 6q^8$

$D_3 = 18q^{10} + 4q^9$

$D_4 = 55q^{12} + 32q^{11} + q^8$

$D_5 = 174q^{14} + 160q^{13} + 30q^{12} + 8q^{10}$

$D_6 = 570q^{16} + 672q^{15} + 332q^{14} + 40q^{12} + 14q^{11}$

$D_7 = 1\,908q^{18} + 2\,712q^{17} + 2\,030q^{16} + 336q^{15} + 168q^{14} + 156q^{13} + 2q^{10}$

$D_8 = 6\,473q^{20} + 10\,880q^{19} + 9\,972q^{18} + 4\,064q^{17} + 869q^{16} + 958q^{15} + 228q^{14} + 22q^{12}$

$D_9 = 22\,202q^{22} + 43\,220q^{21} + 46\,004q^{20} + 27\,392q^{19} + 8\,770q^{18}$
$+ 4\,724q^{17} + 2\,776q^{16} + 164q^{15} + 134q^{14} + 60q^{13}$

$D_{10} = 76\,886q^{24} + 169\,784q^{23} + 207\,444q^{22} + 148\,728q^{21} + 74\,576q^{20}$
$+ 27\,540q^{19} + 18\,816q^{18} + 5\,308q^{17} + 656q^{16} + 728q^{15} + 62q^{14} + 6q^{12}$

$D_{11} = 268\,352q^{26} + 662\,424q^{25} + 912\,378q^{24} + 755\,936q^{23}$
$+ 477\,342q^{22} + 209\,708q^{21} + 107\,490q^{20} + 56\,496q^{19}$
$+ 9\,040q^{18} + 4\,920q^{17} + 2\,000q^{16} + 72q^{14} + 12q^{13}$

$D_{12} = 942\,651q^{28} + 2\,573\,976q^{27} + 3\,923\,948q^{26} + 3\,718\,712q^{25} + 2\,660\,956q^{24}$
$+ 1\,481\,090q^{23} + 696\,836q^{22} + 388\,148q^{21} + 132\,834q^{20}$
$+ 32\,682q^{19} + 21\,268q^{18} + 2\,912q^{17} + 482q^{16} + 378q^{15} + q^{12}$

$D_{13} = 3\,329\,608q^{30} + 9\,967\,932q^{29} + 16\,621\,216q^{28} + 17\,685\,192q^{27}$
$+ 14\,106\,708q^{26} + 9\,040\,536q^{25} + 4\,860\,468q^{24} + 2\,436\,432q^{23}$
$+ 1\,195\,414q^{22} + 352\,580q^{21} + 146\,048q^{20} + 57\,772q^{19}$
$+ 6\,048q^{18} + 3\,980q^{17} + 792q^{16} + 30q^{14}.$

A.3. Honeycomb

(We give $2D_n$ to avoid fractions.)

$2D_1 = 3q^4$

$2D_2 = 6q^5$

$2D_3 = 14q^6$

$2D_4 = 36q^7$

$2D_5 = 93q^8 + 6q^7$

$2D_6 = 244q^9 + 36q^8 + q^6$

$2D_7 = 648q^{10} + 162q^9 + 6q^7$

$2D_8 = 1\,728q^{11} + 660q^{10} + 27q^8$

$2D_9 = 4\,651q^{12} + 2\,394q^{11} + 105q^{10} + 110q^9$

$2D_{10} = 12\,630q^{13} + 8\,172q^{12} + 840q^{11} + 399q^{10} + 33q^9$

$2D_{11} = 34\,566q^{14} + 26\,820q^{13} + 4\,728q^{12} + 1\,362q^{11} + 264q^{10} + 3q^8$

$2D_{12} = 95\,312q^{15} + 85\,572q^{14} + 22\,296q^{13} + 4\,860q^{12} + 1\,488q^{11} + 24q^9$

$2D_{13} = 264\,387q^{16} + 268\,908q^{15} + 92\,322q^{14} + 19\,608q^{13} + 7\,032q^{12}$
$+ 174q^{11} + 135q^{10}$

$$2D_{14} = 736\,974q^{17} + 837\,000q^{16} + 353\,208q^{15} + 84\,234q^{14} + 29\,178q^{13} + 2\,412q^{12} + 636q^{11} + 30q^{10}$$

$$2D_{15} = 2\,062\,784q^{18} + 2\,588\,490q^{17} + 1\,282\,464q^{16} + 361\,958q^{15} + 115\,116q^{14} + 17\,856q^{13} + 2\,631q^{12} + 414q^{11} + 2q^{9}$$

$$2D_{16} = 5\,794\,056q^{19} + 7\,968\,408q^{18} + 4\,494\,144q^{17} + 1\,502\,595q^{16} + 453\,156q^{15} + 101\,136q^{14} + 11\,922q^{13} + 3\,060q^{12} + 27q^{10}$$

$$2D_{17} = 16\,325\,904q^{20} + 24\,432\,360q^{19} + 15\,393\,867q^{18} + 5\,947\,902q^{17} + 1\,824\,759q^{16} + 488\,730q^{15} + 63\,189q^{14} + 17\,310q^{13} + 459q^{12} + 198q^{11}.$$

A.4. Face-centred cubic

$$D_1 = 6q^{22}$$

$$D_2 = 42q^{32} + 24q^{31}$$

$$D_3 = 326q^{42} + 372q^{41} + 192q^{40} + 32q^{39} + 8q^{30}$$

$$D_4 = 2\,739q^{52} + 4\,584q^{51} + 4\,176q^{50} + 1\,896q^{49} + 1\,230q^{48} + 123q^{40} + 120q^{39} + 30q^{38}$$

$$D_5 = 24\,234q^{62} + 53\,640q^{61} + 65\,550q^{60} + 47\,784q^{59} + 35\,880q^{58} + 17\,256q^{57} + 6\,168q^{56} + 384q^{55} + 1\,512q^{50} + 2\,616q^{49} + 1\,728q^{48} + 1\,464q^{47} + 24q^{38} + 12q^{37}$$

$$D_6 = 222\,566q^{72} + 612\,000q^{71} + 934\,992q^{70} + 881\,496q^{69} + 798\,972q^{68} + 534\,336q^{67} + 339\,088q^{66} + 167\,472q^{65} + 33\,960q^{64} + 13\,456q^{63} + 17\,648q^{60} + 41\,298q^{59} + 43\,536q^{58} + 42\,600q^{57} + 25\,080q^{56} + 10\,488q^{55} + 740q^{54} + 504q^{48} + 648q^{47} + 816q^{46} + 2q^{36}$$

$$D_7 = 2\,102\,208q^{82} + 6\,902\,880q^{81} + 12\,582\,594q^{80} + 14\,582\,112q^{79} + 14\,886\,168q^{78} + 12\,568\,800q^{77} + 9\,395\,856q^{76} + 6\,531\,792q^{75} + 3\,379\,560q^{74} + 1\,743\,264q^{73} + 405\,120q^{72} + 202\,440q^{71} + 200\,904q^{70} + 591\,156q^{69} + 809\,040q^{68} + 949\,176q^{67} + 778\,632q^{66} + 580\,704q^{65} + 326\,400q^{64} + 74\,832q^{63} + 32\,568q^{62} + 7\,770q^{58} + 15\,792q^{57} + 23\,088q^{56} + 17\,736q^{55} + 9\,048q^{54} + 744q^{53} + 96q^{46} + 240q^{45}.$$

A.5. Body-centred cubic

$$D_1 = 4q^{14}$$

$$D_2 = 28q^{20}$$

$$D_3 = 204q^{26} + 48q^{25}$$

$D_4 = 1\,562q^{32} + 864q^{31} + 144q^{30} + 12q^{24}$

$D_5 = 12\,544q^{38} + 10\,824q^{37} + 4\,032q^{36} + 960q^{35} + 216q^{30} + 72q^{29}$

$D_6 = 104\,756q^{44} + 120\,048q^{43} + 71\,136q^{42} + 26\,608q^{41} + 7\,344q^{40}$
$\quad + 2\,704q^{36} + 1\,968q^{35} + 696q^{34} + 12q^{28}$

$D_7 = 900\,168q^{50} + 1\,279\,344q^{49} + 1\,001\,412q^{48} + 524\,752q^{47}$
$\quad + 220\,776q^{46} + 57\,024q^{45} + 2\,816q^{44} + 29\,952q^{42} + 34\,440q^{41}$
$\quad + 19\,104q^{40} + 6\,912q^{39} + 312q^{34} + 216q^{33}$

$D_8 = 7\,901\,843q^{56} + 13\,415\,424q^{55} + 12\,729\,888q^{54} + 8\,478\,544q^{53}$
$\quad + 4\,572\,576q^{52} + 1\,873\,584q^{51} + 552\,088q^{50} + 44\,160q^{49}$
$\quad + 318\,594q^{48} + 483\,684q^{47} + 373\,584q^{46} + 206\,352q^{45} + 65\,712q^{44}$
$\quad + 3\,768q^{43} + 5\,262q^{40} + 5\,808q^{39} + 3\,072q^{38} + 27q^{32}$

A.6. Simple cubic

$D_1 = 3q^{10}$

$D_2 = 15q^{14}$

$D_3 = 83q^{18} + 12q^{17}$

$D_4 = 486q^{22} + 192q^{21} + 3q^{16}$

$D_5 = 2\,967q^{26} + 1\,992q^{25} + 270q^{24} + 48q^{20}$

$D_6 = 18\,748q^{30} + 17\,616q^{29} + 5\,700q^{28} + 400q^{27} + 496q^{24} + 126q^{23}$

$D_7 = 121\,725q^{34} + 145\,872q^{33} + 73\,902q^{32} + 16\,104q^{31} + 384q^{29}$
$\quad + 4\,368q^{28} + 2\,676q^{27} + 264q^{26} + 18q^{22}$

$D_8 = 807\,381q^{38} + 1\,173\,216q^{37} + 785\,448q^{36} + 299\,472q^{35} + 29\,280q^{34} + 9\,216q^{33}$
$\quad + 36\,027q^{32} + 34\,782q^{31} + 10\,764q^{30} + 408q^{28} + 378q^{26} + 72q^{25}$

$D_9 = 5\,447\,203q^{42} + 9\,296\,964q^{41} + 7\,608\,912q^{40} + 3\,986\,592q^{39}$
$\quad + 970\,845q^{38} + 167\,760q^{37} + 321\,756q^{36} + 370\,032q^{35} + 201\,768q^{34}$
$\quad + 25\,212q^{33} + 9\,792q^{32} + 4\,854q^{30} + 2\,892q^{29} + 212q^{27} + 8q^{24}.$

A.7. Diamond

$D_1 = 2q^{6}$

$D_2 = 6q^{8}$

$D_3 = 22q^{10}$

$D_4 = 91q^{12}$

$D_5 = 396q^{14} + 12q^{13}$

$D_6 = 1\,782q^{16} + 144q^{15} + 2q^{12}$

$D_7 = 8\,186q^{18} + 1\,248q^{17} + 24q^{14}$

$D_8 = 38\,199q^{20} + 9\,120q^{19} + 192q^{18} + 207q^{16}$

$D_9 = 180\,544q^{22} + 60\,504q^{21} + 3\,318q^{20} + 128q^{19} + 1\,508q^{18} + 60q^{17}$

$$D_{10} = 862\,642q^{24} + 377\,520q^{23} + 37\,836q^{22} + 2\,048q^{21} + 9\,978q^{20} + 1\,038q^{19} + 60q^{18} + 6q^{16}$$

$$D_{11} = 4\,161\,378q^{26} + 2\,259\,888q^{25} + 348\,966q^{24} + 26\,664q^{23} + 62\,112q^{22} + 11\,832q^{21} + 960q^{20} + 102q^{18} + 12q^{17}$$

$$D_{12} = 20\,245\,844q^{28} + 13\,148\,256q^{27} + 2\,820\,900q^{26} + 294\,464q^{25} + 379\,065q^{24} + 109\,086q^{23} + 12\,384q^{22} + 1\,146q^{20} + 192q^{19} + q^{16}$$

$$D_{13} = 99\,248\,728q^{30} + 74\,993\,100q^{29} + 20\,851\,026q^{28} + 2\,896\,056q^{27} + 2\,316\,828q^{26} + 886\,860q^{25} + 135\,720q^{24} + 4\,800q^{23} + 10\,434q^{22} + 2\,436q^{21} + 16q^{18}.$$

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